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On Birkhoff Interpolation: Free Birkhoff Nodes

H. WINDAUER

Rechenzentrum, Ruhr-Universität Bochum, West Germany

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1. INTRODUCTION

The elements e_{mn} of an $M \times (N+1)$ incidence matrix $E = (e_{mn})$, $m = 1,..., M$, $n = 0,..., N$, are 0 or 1. We assume that the set e of all pairs (m, n), for which $e_{mn} = 1$, has $N + 1$ elements. The Birkhoff interpolation problem [2] associated with E is to find a polynomial p of degree (at most) N that satisfies the conditions

$$p^{(n)}(x_m) = a_{mn}, \qquad (m, n) \in e, \tag{1.1}$$

where the $a_{mn} \in \mathbb{R}$, $(m, n) \in e$, and the real points

$$x_1 < x_2 < \cdots < x_M \tag{1.2}$$

are given.

Following G. G. Lorentz and K. L. Zeller [4], a matrix E is called free, if the Birkhoff interpolation problem (1.1) has a solution for each selection of the points (1.2) and of the real data a_{mn}. We shall call E conditionally free, if there are points (1.2) such that (1.1) has a solution for each selection of the real data a_{mn}.

As I. J. Schoenberg [5] has shown, an $M \times (N+1)$ incidence matrix E can only be free, if it satisfies the "Pólya condition"

$$M_n = \sum_{k=0}^{n} m_k \geqslant n + 1, \qquad n = 0,..., N, \tag{1.3}$$

where $m_k = \sum_{m=1}^{M} e_{mk}$. D. Ferguson [3] proved that E is conditionally free if and only if (1.3) holds. Independently, K. Atkinson and A. Sharma [1] and D. Ferguson [3] found certain sufficient conditions for E to be free.

But there are many simple matrices like

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix} \quad \text{or} \quad \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix},$$

which are only conditionally free. In order to find suitable points (1.2) for conditionally free matrices, we shall introduce the idea of free Birkhoff nodes.

2. Free Birkhoff Nodes

In the following, let Π_N be the real vector space spanned by the mappings $\pi_n: \mathbb{R} \ni x \mapsto x^n$, $n = 0,..., N$.

We shall call a set of M real points (1.2) free, if the Birkhoff interpolation problem (1.1) associated with any conditionally free $M \times (N+1)$ incidence matrix E has a solution $p \in \Pi_N$ for each selection of the real data a_{mn}. Such points (1.2) will also be called free Birkhoff nodes.

Let be $\mathbb{R} \ni c > \frac{1}{2}$, $N \in \mathbb{N}$ and $f_1, f_{-1} \in \mathbb{R}$ with

$$f_1, f_{-1} \geqslant \max\{(c - \tfrac{1}{2})^{-1}, (4N^2)^{N+1}\},$$

and $f_i, f_{-i} \in \mathbb{R}$ with $f_i \geqslant f_{i-1}$, $f_{-i-1} \geqslant f_{-i}$, $i = 2, 3,...$. Further, let $\mathscr{F}$ be the set of all points $x_{\pm i}$, $i = 1, 2, 3,...$, defined by

$$x_{-i} = -c + (f_{-i})^{-i}, \qquad x_i = c - (f_i)^{-i}, \qquad i = 1, 2, 3,... .$$

Theorem. *Each ordered subset $\mathscr{M} \subset \mathscr{F}$ with M elements ($M \in \mathbb{N}$) is free.*

Proof. Let E be an arbitrary conditionally free $M \times (N+1)$ incidence matrix. We shall prove that any $p \in \Pi_N$ vanishes identically if

$$p^{(n)}(x_m) = 0, \qquad x_m \in \mathscr{M}, \quad (m, n) \in e.$$

Suppose that $M_n - n$ zeros of $p^{(n)}$, $0 \leqslant n \leqslant N-1$, are among the points of $\mathscr{M}$ or belong to the open intervals

$$I_0^{(n)} = (y_{-1} + d^n, y_1 - d^n), \qquad I_i^{(n)} = (y_i, y_{i+1} - (f_i)^{-i} \cdot d^n),$$

$$I_{-i}^{(n)} = (y_{-i-1} + (f_{-i})^{-i} \cdot d^n, y_{-i}),$$

where $d = (4 \cdot N^2)^{-1}$ and the $y_{\pm i}$ are the elements of $\mathscr{F}$, $i = 1, 2, 3,...$. This is true for $n = 0$. By Rolle's theorem, we get $M_n - n - 1$ zeros of $p^{(n+1)}$. Similar to the proof of Lemma 2 of G. G. Lorentz and K. L. Zeller [4], these zeros can be selected to be distinct and out of the intervals $I_0^{(n+1)}$, $I_{\pm i}^{(n+1)}$. E also specifies m_{n+1} new zeros of $p^{(n+1)}$ in m_{n+1} points of $\mathscr{M}$. Thus $p^{(n+1)}$ must have at least $M_n - n - 1 + m_{n+1} = M_{n+1} - (n+1)$ zeros. Now it follows by induction on n that $p^{(n)}$ has at least $M_n - n$ zeros, $n = 0,..., N$; this, and the Pólya condition (1.3), assures that $p^{(N)}$ has at least $M_N - N =$

$N + 1 - N = 1$ zeros. But $p^{(N)} = c \cdot \pi_0$, $c \in \mathbb{R}$. Thus $p^{(N)} = 0$ and $p \in \Pi_{N-1}$. These arguments work for all derivatives $p^{(n)}$, $n < N$, and guarantee that p vanishes identically.

Obviously, any affin transformation maps free Birkhoff nodes into free Birkhoff nodes again. In this way, free Birkhoff nodes can be constructed in any interval $[a, b] \subset \mathbb{R}$.

References

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